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Intermediate-Complexity Biological Modeling Framework for Nutrient Cycling in
Lakes Based on Physical Structure

by

Michael Clay Rigley

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of the requirements for the degree

of

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An Intermediate-Complexity Biological Modeling Framework for Nutrient Cycling in Lakes Based on Physical Structure

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Key Words: total dissolved nitrogen, seston, nutrient cycling, compartment model, plunge, Bull Trout Lake, multi-compartment model

Abstract

Mathematical models for the change in concentration of total dissolved nitrogen (TDN) in mountain lakes are developed based on the dynamics of coupled, well-mixed containers. Each includes a stratified lake structure without the complexity of a full fluid model. A lake is divided into a suite of compartments based on physical structure: warm upper layer (epilimnion), cold inflow and insertion layer (metalimnion), cold lower layer (hypolimnion), and a warm shallow shelf. With the compartments as the framework and literature values for uptake rates, death rates, half-saturation constants, and sinking rates, systems of equations are written for three models. The first is a system of differential equations including nutrient cycling within each compartment, including changes in TDN due to growth and death of seston as well as loss to and gain from lake sediments. With the literature values, flows, and TDN data taken at the inflow (Baker and Wurtsbaugh 2008), these equations are solved numerically to determine the concentration of

TDN in each compartment. The second is a simplified version of the first model containing only fluxes of TDN between compartments and the flux in lake sediments. The third is a system of equations for the steady states of the first model found by making an assumption on the half-saturation constant. With TDN data taken at the outflow in 2002 (Baker and Wurtsbaugh 2008), terms of sedimentation fluxes are chosen to minimize the sum of the squared difference between the measured and predicted concentrations of TDN for each of the three systems. Each model is tested against data taken in 2003 (Baker and Wurtsbaugh 2008). The second model predicts observed TDN well in a stratified lake structure without the computational difficulty of a full fluid model. To determine the flows between compartments we solved differential equations for the transport of nutrients into lakes by cold plunging inflows (Hauenstein and Dracos 1984). The entrainment rates are treated as eigenvalues, and an eigenvalue problem is solved for the plunging inflow based on discharge data taken in 2003 (Arp 2006) and data taken from a field study on Bull Trout Lake (BTL) in central Idaho in June 2008. The results show that lake structure is a significant factor in relating input and output concentrations of TDN.

1 Introduction

Mountain lakes play an important role in ecosystem function and management. As major stopping points for transport of nutrients and pollutants in a drainage basin (i.e. watershed), lakes are natural accumulators and bio-reactors. They play a significant role in uptake and distribution of nutrients in the run off, primary production by phytoplankton at the base of the food chain, and check points for water quality. Lakes attenuate nutrient pulses, serving some times as nutrient sources, other times as sinks (Wurtsbaugh et al. 2005). From these mixing pots a certain amount of the nutrients and pollutants flow downstream affecting further ecosystems and entering public water supplies. As charismatic and accessible landscape features, mountain lakes are a touchstone for recreation and public perception of ecosystem health. Use of models to predict nutrient transport in these lakes can therefore impact a variety of research enterprises, from basic biological and ecosystem processes to water management and climate scenario exploration.

Although more attention is given to man-made reservoirs and oceans, in recent years, natural lakes have garnered more attention due to some of their unique hydrological dynamics (Fleenor 2000). Cold mountain input streams have a higher density than warmer lake water, and therefore plunge beneath the ambient lake water. This higher density current continually loses density as it entrains ambient lake water and eventually becomes neutrally buoyant with the lake water at which point the current inserts into the middle layer of the lake (Fleenor 2000). In this paper we address the question of the significance of mountain lake structure on the nutrient cycling through mountain lakes and the relative effect of biological activity on nutrients cycling through lakes.

Several models have been created for the nutrient cycling within natural lakes. The full complexity of fluid motions within a lake can be predicted using the Navier-Stokes equations with Boussinesq (NSB) approximation for buoyancy. Nutrient concentrations and population processes can be treated as Lagrangian variables carried with the flow (see, for example, DYRESM at the University of Western Australia or the Hydrodynamic and Water Quality Model at Portland State University (Imberger 1981)). Obtaining the solution of the NSB is computationally intensive and data requirements at the boundaries (lake bottom topography, surface wind forcing and radiative flux) are large. Such models are framed as partial differential equations and biological parameters are treated as continuous functions of temperature and flow parameters, vastly complicating parameter estimation.

Simpler models include a single well-mixed container, essentially a chemostat (Wurtsbaugh 2005). Unfortunately most lakes are not well-mixed; solar input warms the upper layers (epilimnion), creating a vertical stratification of temperature and inhibiting vertical mixing. Cold, dense mountain input streams plunge beneath the warmer lake water. The higher density current continually loses density as it entrains lake water and eventually becomes neutrally buoyant, at which point the current inserts into the middle layer, metalimnion, of the lake (Fleenor 2000). Vertical mixing then occurs between the metalimnion and the epilimnion. At Bull Trout Lake (BTL) in Central Idaho part of the flow rises into the epilimnion where it crosses a shallow shelf, where nutrients are lost and gained from lake sediments before

proceeding to the outflow. Flows within the lake are not uniform; variable and cascading delays are created between the lake's inflow and outflow. A map of this structure is given in Figure 1 below.

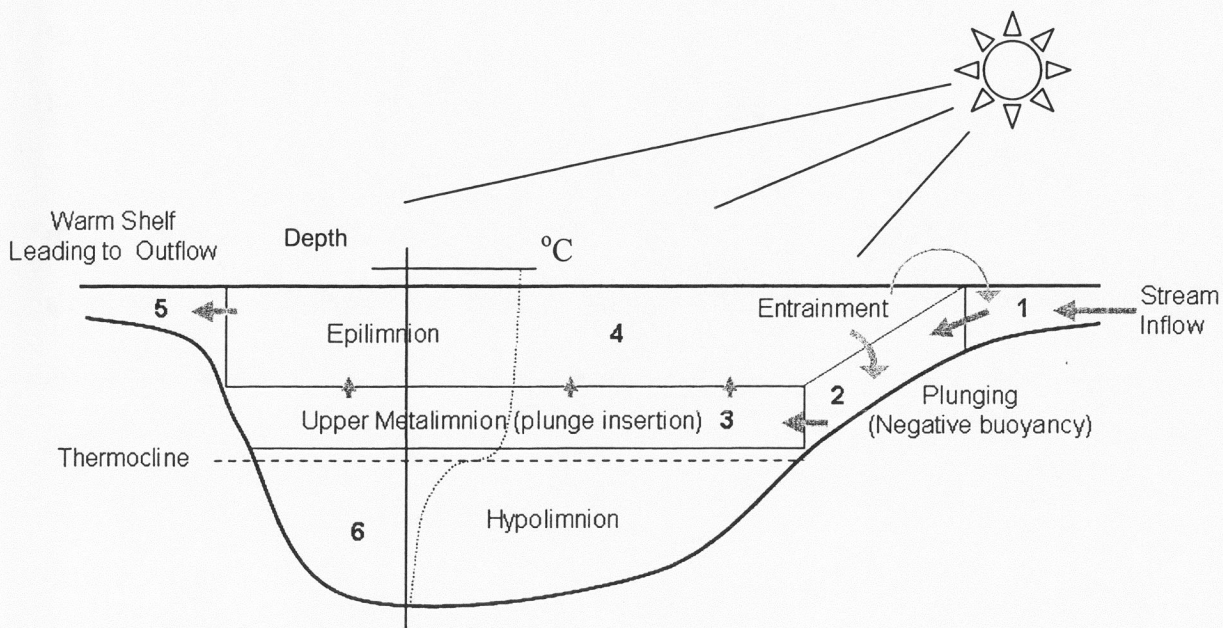


Figure 1. Warming from solar input and cold inflows create a stratified lake structure.

These hydrological dynamics complicate our understanding of lake processes including plankton growth rates which depend on gradients in light, temperature and residence time. One-container chemostat models are not satisfying in such circumstances because, even though the equations modeling the biological processes of a single well-mixed container are well understood, they cannot describe the dynamics associated with a stratified lake structure.

We present three lake models that use a suite of spatial compartments in relating the input and output values of nutrients, namely TDN. This compartment structure avoids the complexity of a NSB model and the simplicity of a single chemostat. These intermediate complexity models include the stratified lake structure, the dynamics of the plunge of cold mountain inflows, nutrient uptake in the warmer upper layers, and seepage through lake sediments. The compartments are determined by the thermal and physical structures of the lake and by the different biological processes within the lake; for example, many lakes have a wide warm shelf that covers a broad area around the deeper colder regions of the lake. Thus the number and particular biological processes of compartments may vary from lake to lake.

The study lake, BTL, like many mountain lakes, has a single large input stream which carries nutrients and plunges below the surface of the lake. The inflow, while entraining some lake water, flows over a small delta that extends into the lake until it plunges down a steep slope at the end of the delta, picking up both vertical and horizontal entrainment from the ambient lake water.

For BTL a suite of three compartments is constructed. The first compartment is made up of the upper regions of the lake or epilimnion; it is assumed to be about 2m deep based on the warm shelf that extends around the lake; the second compartment is determined by the dynamics of the inflow—the end of the plunge is assumed to be near the sharp gradient in the stratification of the temperature of the lake (thermocline). The thermocline was observed at varying depths (between 2 and 6m) at different times during the day, so compartment 2 is determined by taking the upper and lower bounds observed for the thermocline; there is assumed to be no vertical mixing with the lower layer (hypolimnion) of the lake (below 6m) where little growth occurs due to lack of sunlight; compartment 3 represents the shallow shelf at BTL where seepage from sediments, groundwater, and other small mountain inlets are included.

For this compartment structure three models are created and tested. Each predicts the concentration of total dissolved nitrogen (TDN) at the outflow given the concentration at the inflow. Total dissolved nitrogen consists of dissolved inorganic nitrogen, including ammonium (NH_4^+), nitrate (NO_3^-) and nitrite (NO_2^-), and dissolved organic nitrogen, including amino acids, proteins, urea, and humic and fulvic acids (Bronk 2000).

The first model is a biological model including the nutrient cycling between total dissolved nitrogen and seston. Seston is defined as minute particulate material moving in water that is composed of both living organisms, such as plankton, and non-living matter such as plant debris and soil particles (NOAA's Coral Reef Information System, <<http://coris.noaa.gov/glossary/glossary_1_z.html>>). The equations for each compartment include growth and death rates, half saturation constants, sinking rates, and flux rates in lake sediments. The concentrations of TDN and seston within each of the several compartments are connected by fluxes based on the flow rates between the compartments, thus creating a coupled system of equations for the entire lake. With input values of initial concentration of TDN, seston, and literature

values for growth rates, death rates, half-saturation constants and sinking rates this system of differential equations is solved numerically to determine the concentration of TDN and seston in each compartment in terms of time.

The second model is a model of the flux of TDN between the lake compartments independent of seston, but still including terms of flux from lake sediments. This model is formed by removing the terms of seston from the first model. This simpler system of differential equations is also solved numerically given input data of the flows between compartments and initial concentrations of TDN.

The third model is a model of the steady states of the second model. Setting the changes in concentration of TDN to zero, the second model becomes a homogeneous system of linear equations which is solved by row-reduction and back substitution for the steady states. This steady state analysis captures a general relationship between inflow and the outflow in terms of the concentration of TDN from lake sediments.

In each model, the terms of flux in lake sediments are determined by minimizing the sum of the squares of the differences between measured and predicted concentration values of TDN. The terms of flux in sediments are determined using data taken in 2002 (Baker and Wurtsbaugh 2008). The resulting models are then tested against data taken in 2003 (Baker and Wurtsbaugh 2008).

Given the optimal seepage and entrainment rates, each model predicts the concentration of TDN at the outflow of the lake. Each model allows the possibility of a stratified lake structure and the effects from seepage from lake sediments and other mountain inlets without the complexity of a full fluid model. The models are of varying levels of complexity from a simple linear system to a non-linear system of differential equations. The results show that the lake structure is significant in determining the cycling of TDN within the lake.

The hydrological dynamics of the lake are represented by the integral models of *momentum* and *buoyancy-dominated* regions as constructed by Hauenstein and Dracos (1984). When a stream first enters a lake, the force of the stream displaces the lake water near the inflow while entraining some lake water. This is called the *momentum-dominated* region. After entering the lake the density differences

between the stream and lake waters, and the slope of the lake bed cause the inflow to plunge beneath the warmer lake water. This is defined as the *buoyancy-dominated* region. We use this basic mathematical framework to determine the flows between compartments in the three models, and the amount of nutrients being cycled within the lake through entrainment.

We solve the differential equations of the momentum-dominated region analytically, while the equations for the buoyancy-dominated region are solved numerically to determine the height (h), breadth (b), and velocity (u) of the plunge in terms of the distance from the inflow. The flow rates ($q = ubh$) between the compartments of the lake are then determined assuming conservation of volume. These flow rates between compartments are the backbone of the three compartment models.

To test the flow model, measurements were taken as part of a field study at BTL in June 2008. The initial height, width, and breadth of the inflow, some measurements near the insertion point, and general observations were made to fit the flow model. With these measurements, the horizontal and vertical entrainment rates are treated as eigenvalues and are used to match the initial conditions of the height, breadth and velocity of the plunge. The flows from the fit model are then used in the three compartment models.

2 Compartment Model

2.1 Compartments

To predict the flow of nutrients through the lake, we choose compartments to isolate particular biological processes to specific parts of the lake. To determine the compartments several factors are considered. Does the lake morphometry suggest natural compartments like shallow or deeper shelves, deeper regions, or sharp inclines? Does the thermal structure based on sunlight naturally separate warmer inhabitable regions from colder, uninhabitable regions? Do the flows near the inflow or outflow or other lake inlets suggest any currents or structure?

The morphometry of Bull Trout Lake (Figure 2) does show a shallow shelf (0-2m) that extends all around the lake, especially on the northwestern side and near the outflow on the northeast side. There is a

sharp incline near the inflow at the end of the delta. The thermal structure of the lake was measured throughout the lake and the thermocline was calculated at different depths at different points and at different times during the day. Near the inflow, the thermocline was observed to vary between about 2m and 6m. The discharge at the outflow was often measured to be greater than the discharge at the inflow suggesting a significant amount of water entering the lake from sources other than the inflow. There are numerous small unmeasured inflows around the lake as well as groundwater coming through the lake bed.

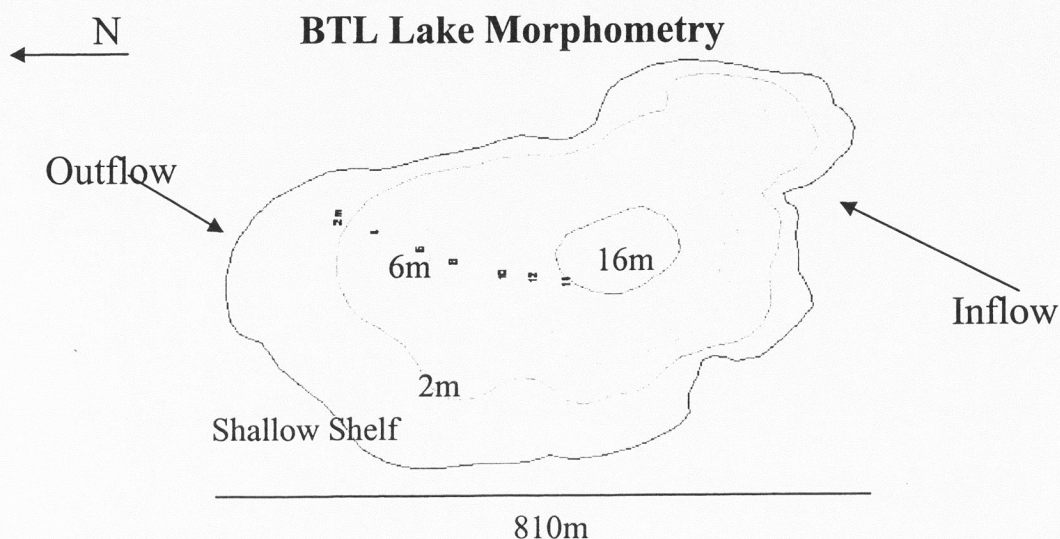


Figure 2. Contour map of BTL. The lake flow from south to north, has a large shallow shelf along the northwestern side of the lake, a delta that runs about 23 meters into the lake, and a maximum depth of about 16 meters.

All of these factors are considered in determining the compartments to BTL. A three compartment model is formed based on the above observations. The first compartment, from the surface to the shallow limit of the thermocline, is chosen to be 2m deep. It includes the volume of water not intersecting the lake bottom, except along the eastern and western boundaries. The second compartment is determined by the measurements of the plunge near the inflow. The thermocline depth was measured to vary from 2 to 6 meters throughout the day, so the second compartment is determined as the region of the lake between 2 and 6 m. This compartment can be thought of as a mixing layer or the layer into which most of the stream sediment initially enters before sinking or rising to other compartments. The third compartment is

the shallow shelf that extends from the northwestern side of the lake up near the outflow. Due to the shallow depth, this region is much warmer than the deeper areas of the lake. Also, as flows move over this shallow shelf, nutrients are released from sediments and carried to the outflow. The region of the lake below 6m could be termed a fourth compartment, but due to the depth and lack of sunlight, it is assumed to have little interaction with the upper compartments. A depiction of the compartmentalization is presented in Figure 3.

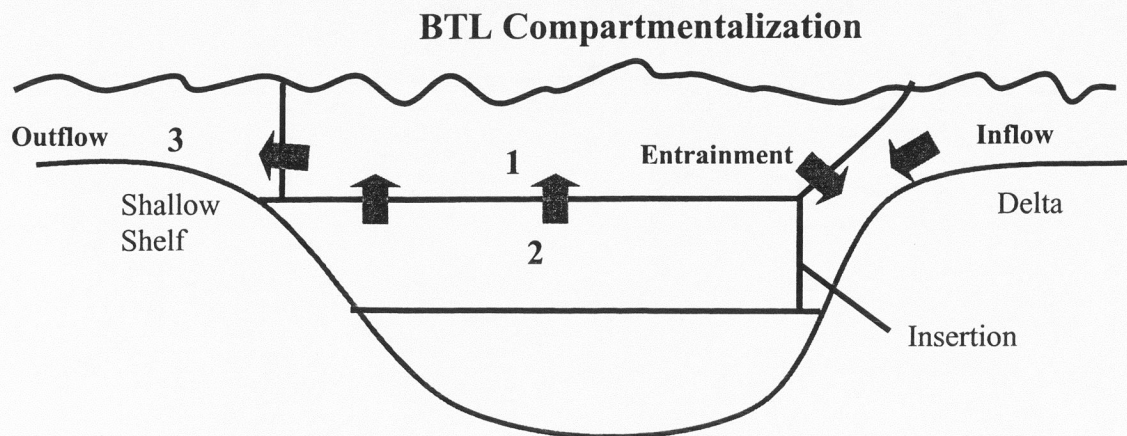


Figure 3. Graphical depiction of compartmental structure. The 3 compartments for BTL are: 1- Epilimnion; 2-Metalimnion or insertion layer; 3-Shallow Shelf to the outflow. Arrows are indicative of hydrological flows between compartments.

With these compartments determined, particular biological processes are isolated to particular compartments within the lake. More growth occurs in compartments 1 and 3 where the most sunlight reaches. More movement occurs in compartment 2 where the plunge inserts and where nutrients mix with the other compartments in the lake. Each compartment is considered as a separate well-mixed container and the transport between them is determined from the flows in the flow model.

2.2 Flows Between Compartments

The volumes of the compartments vary seasonally as the lake fluctuates between a well-mixed state in the spring and fall and a stratified state in the summer and winter. These changes can be considered relative to the season of the year and the volume can be assumed constant for a given time period. As can be seen in Figure 3 above, with the three compartment structure there are four boundaries between compartments,

making 4 flow rates between compartments: the inflow rate (q_{02}), connecting the input stream with compartment 2; the entrainment rate from compartment 1 into compartment 2 (q_{12}); the vertical flow rate from compartment 2 to compartment 1 ($q_{21} = q_{02} + q_{12}$); and the flow from compartment 1 to compartment 3 ($q_{13} = q_{02}$). The outflow is the same value as q_{02} since the volume in compartment 3 must be conserved.

These flow rates determine the flux of nutrients between the three compartments of the lake. Since the volume of each compartment is conserved, the measured outflow on each day is used as the inflow to account for water from sources other than the inflow. The flow rates for the model are determined by solving the system of equations of Hauenstein and Dracos (HDE). The solution for the flow model is given in Section 3.

2.3 Definition of Variables

We now treat each compartment as a well-mixed container, connected by the flow rates. For each model, equations are written for each compartment and the equations are connected by the flow rates between them. The terms are defined in relation to the first model and the second and third models are then derived from the first model.

For each compartment ($i = 1, 2, 3$) we have the concentration of seston (S_i) and TDN (N_i) in mg/m^3 , the volume (V_i) and bottom surface area (A_i) in m^3 and m^2 respectively, half saturation constants for TDN uptake by seston (a_i) in mg/m^3 , growth rates (μ_i), death rates (d_i), sinking velocities (σ_i) in m/day , velocities at which nutrients are released from lake sediments in each compartment (α_i) in m/day , TDN flux from sediments in each compartment ($R_i = \alpha_i N_i^s A_i$) in mg/day , and the flow rates between compartments (from compartment i to j , q_{02} , q_{21} , q_{12} , q_{13}). A list of these parameters is given below (Table 1).

2.4 Model 1 - Biological Compartment Model

With the above parameters we have the equations for the first model:

Symbols	Description [for $i = 1, 2, 3$] (units)
N_i	concentration of TDN (mg/m^3)
N_o	inflow concentration of TDN (mg/m^3)
S_i	concentration of Seston (mg/m^3)
V_i	volume of compartment (m^3)
A_i	bottom surface area of compartment (m^2)
a_i	half saturation constant (mg/m^3)
μ_i	growth rate of Seston (1/day)
d_i	death rate of Seston (1/day)
σ_i	sinking velocity (m/day)
N_i^s	concentration in the sediment of compartment i (mg/m^3)
α_i	uptake velocity of sediment in compartment i (m/day)
R_i	flux from sediments in each compartment (mg/day)
q_{02}	flow rate of inflow (m^3/day)
q_{12}	flow rate from entrainment (m^3/day)
q_{21}	vertical flow rate (m^3/day)
q_{13}	flow rate at outflow (m^3/day)

Table 1. List of variables and constants used in compartment biological model.

$$(V_1 S_1)' = \left(\frac{\mu_1 S_1 N_1}{a + N_1} V_1 \right) + q_{21} S_2 - q_{12} S_1 - q_{13} S_1 - \sigma A_1^w S_1 - d S_1 V_1 \quad (1)$$

$$(V_1 N_1)' = - \left(\frac{\mu_1 S_1 N_1}{a + N_1} V_1 \right) + q_{21} N_2 - q_{12} N_1 - q_{13} N_1 + d S_1 V_1 + R_1 \quad (2)$$

$$(V_2 S_2)' = \frac{\mu_2 S_2 N_2}{a + N_2} V_2 - d S_2 V_2 - \sigma A_2^w S_2 + \sigma A_1^w S_1 + q_{12} S_1 - q_{21} S_2 \quad (3)$$

$$(V_2 N_2)' = - \left(\frac{\mu_2 S_2 N_2}{a + N_2} V_2 \right) + d S_2 V_2 + q_{12} N_1 - q_{21} N_2 + q_{02} N_o + R_2 \quad (4)$$

$$(V_3 S_3)' = \left(\frac{\mu_3 S_3 N_3}{a + N_3} V_3 \right) + q_{13} S_1 - q_{13} S_3 - d S_3 V_3 - \sigma A_3 S_3 \quad (5)$$

$$(V_3 N_3)' = - \left(\frac{\mu_3 S_3 N_3}{a + N_3} V_3 \right) + q_{13} N_1 - q_{13} N_3 + d S_3 V_3 + R_3 \quad (6)$$

This model includes terms relating the concentrations of TDN and seston. We have terms for the growth

and death of seston from TDN in terms of the half-saturation constant a_i and growth rate $\mu_i \left(\frac{\mu_i S_i N_i}{a_i + N_i} V_i \right)$

and $d_i S_i V_i$ respectively); terms of flux due to sinking ($\sigma_i A_i^w S_i$); terms of flux of nutrients into and from lake sediments (R_i); and fluxes between compartments ($q_{ji} N_j$) (Figure 4).

Model 1 – Biological Model

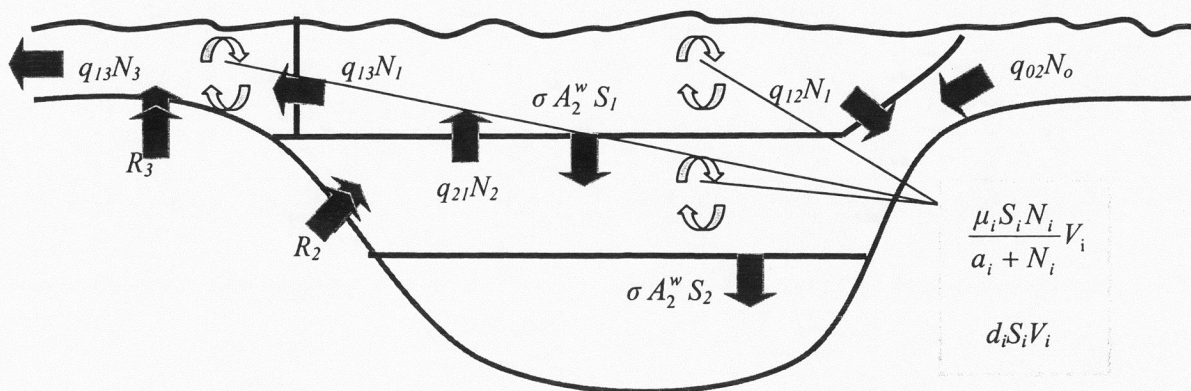


Figure 4. Graphical representation of terms in biological compartment model. Each term represents change in mass per time. For each compartment we have equations for the change in concentration of seston and the change in concentration of TDN:

We used Axler and Reuter (1996) to estimate the half saturation constant, $a = 11.3 \text{ mg/m}^3$, assumed to be uniform throughout all compartments; the sinking rate was estimated from Changsheng Chen (2002) et al, $\sigma = .6 \text{ m/day}$; and the death rate was estimated from Bonnet (2001), $d = .05 \text{ /day}$, both assumed to be uniform throughout all compartments.

The volume of each compartment and bottom surface areas are taken from data previously collected at that site (Wurtsbaugh 2008). The surface area at various depths is given directly. At the lake surface the area is given as 0.28 km^2 . At two meters, the area of the lake is given as 0.17 km^2 . This is the area of compartment 1 with a boundary of lake water ($A_1^w = 0.17 \text{ km}^2$). The difference of this and the original area is the approximate area between 0 and 2 meters contacting the lake bottom ($0.28 - 0.17 = 0.11 \text{ km}^2$). Of this, approximately two thirds of the area is on the shallow shelf region or compartment 3 ($A_3 = 0.074 \text{ km}^2$), and the remaining area is the boundary of compartment 1 with the lake sediment ($A_1^s = 0.036 \text{ km}^2$). The area at the bottom boundary of compartment 2 is given as 0.09 km^2 or

($A_2^w = 0.09 \text{ km}^2$). The difference of A_2^w and A_1^w is the approximate surface area between 2 and 6 meters contacting the lake bottom ($A_2^s = 0.17 - 0.09 = 0.08 \text{ km}^2$). Compartment 3 has no boundary that does not contact lake sediment.

The volumes are given as the amount of water above a particular depth. Thus to find the volume of any given compartment, the volumes from the depths above are subtracted. The volume above 2 meters is $4.6 \times 10^5 \text{ m}^3$. Of this, approximately one third of the volume is the shallow shelf ($V_3 = 1.52 \times 10^5 \text{ m}^3$), and the remainder is compartment 1 ($V_1 = 3.08 \times 10^5 \text{ m}^3$). The volume above 6 meters is $9.55 \times 10^5 \text{ m}^3$. The difference of this and the volume at 2 meters gives the volume of the second compartment ($V_2 = 4.95 \times 10^5 \text{ m}^3$).

The initial concentrations of TDN and seston in each compartment are taken from data in 2002 (Baker and Wurtsbaugh 2008). The values for the flows between compartments are calculated daily from the solution of the flow model. The product ($R_i = \alpha_i A_i^b N_i^s$) of the velocity of seepage in sediments (α_i), the area of each compartment intersecting the lake bottom (A_i^b), and the concentrations of TDN in the sediments (N_i^s) will be determined for each model by minimizing the sum of the squares of the difference between the measured and predicted values of TDN at the outflow using the MATLAB function, `fminsearch`. For a complete list of parameters in each compartment model see Table 2 below.

With the above conditions the non-linear system of equations for the first model is solved numerically for the concentrations of seston and TDN in each compartment, S_i and N_i respectively ($i = 1, 2, 3$), in terms of time assuming that the compartment volumes remain relatively constant. The system is put into matrix form and solved using a MATLAB solver.

From this we obtain the concentrations of TDN and seston in each compartment over the given interval. The interval is the amount of time between values of measured TDN at the inflow. Of particular interest is the concentration of TDN at the outflow (N_3), which can be compared with observations.

Symbols	Description
$V_1 = 3.08 \times 10^5 \text{ m}^3$	volume of compartment 1
$V_2 = 4.95 \times 10^5 \text{ m}^3$	volume of compartment 2
$V_3 = 1.52 \times 10^5 \text{ m}^3$	volume of compartment 3
$A_1^w = 0.17 \text{ km}^2$	bottom surface area of compartment 1
$A_1^s = 0.036 \text{ km}^2$	surface area of compartment 1 with lake sediment
$A_2^w = 0.09 \text{ km}^2$	bottom surface area of compartment 2
$A_2^s = 0.08 \text{ km}^2$	surface area of compartment 2 with lake sediment
$A_3^s = 0.074 \text{ km}^2$	surface area of compartment 3 with lake sediment
$a = 11.3 \text{ mg/m}^3$	half saturation constant
$\mu_1 = .6 \text{ per day}$	growth rate of Seston in compartment 1
$\mu_2 = .3 \text{ per day}$	growth rate of Seston in compartment 2
$\mu_3 = .6 \text{ per day}$	growth rate of Seston in compartment 3
$d = .05 \text{ per day}$	death rate of Seston
$\sigma = .6 \text{ m/day}$	sinking velocity

Table 2. List of variables and constants used in the biological compartment model.

2.5 Flux of TDN in Lake Sediments

We have three terms in the biological model not determined by initial conditions or calculated as variables, namely the terms of the flux of TDN in the sediments (R_1, R_2, R_3). We find R_1, R_2 , and R_3 that fit the solution of the biological model to the actual measured values of TDN at the outflow. This is done by taking the sum of the squares of the errors between the actual and predicted values of TDN on given Julian days (JD) in 2002.

$$\varepsilon^2 = \sum_{i=1}^n (N_3(R_1, R_2, R_3) - N_{actual}(Day = i))^2$$

where n is the number of days used.

The minimum of the sum above is found by using the same MATLAB function `fminsearch`. A function for the solution to the first model is written. The input values are the initial values of the concentration of TDN and seston for each compartment and a variable vector representing the three sediment terms and the output is the sum above. The matlab function `fminsearch` finds the optimal choices of R_1, R_2 , and R_3 to minimize the sum. The initial inflow and outflow values of TDN were taken from BTL data in 2002 to find the optimal choices and were then tested against data in 2003 (Baker and Wurtsbaugh 2008).

2.6 Model 2 - TDN Model

The second model is a simplified version of the first model. Analysis of the first model suggested that the concentration of TDN is largely independent of the concentration of seston. Removing the terms of seston from the system, we have a system of three equations for the concentration of TDN in each compartment based on the flows between compartments and the flux of nutrients in lake sediments.

$$(V_1 N_1)' = q_{21} N_2 - q_{12} N_1 - q_{13} N_1 + R_1 \quad (7)$$

$$(V_2 N_2)' = q_{12} N_1 - q_{21} N_2 + q_{02} N_o + R_2 \quad (8)$$

$$(V_3 N_3)' = q_{13} N_1 - q_{13} N_3 + R_3 \quad (9)$$

This system is solved numerically for the concentrations of TDN in each compartment in terms of time. The same code is used as the one above with the seston terms removed. The terms of the flux of the concentration of TDN in the sediment (R_1 , R_2 , and R_3) are recalculated for this new system using the same method as in section 2.5, that is they are used to fit the data from 2002 and tested against the data in 2003.

2.7 Model 3 - Steady States

The third model is the steady states of the first model with an assumption on the half-saturation constant (a). The half-saturation constant (a) in the first model was assumed to be much smaller than the concentration of TDN (N_i) in each compartment; this changes the nonlinear term of growth between TDN and seston to a linear term:

$$\frac{\mu_i S_i N_i}{a_i + N_i} V_i \rightarrow \mu_i S_i V_i.$$

The biological model turns into a linear system of differential equations. To solve for the steady states we set the changes in concentration of TDN and Seston in each compartment to zero. This creates a homogeneous system of linear equations. This linear system is solved analytically by row reducing and using back substitution to find the steady state solution:

$$S_1 = S_2 = S_3 = 0$$

$$N_1 = \frac{q_{02}N_o + R_2 + R_1}{q_{13}}, N_2 = \frac{q_{02}N_o q_{12} + q_{02}N_o q_{13} + R_2 q_{12} + R_2 q_{13} + R_1 q_{12}}{q_{21}q_{13}}, N_3 = \frac{q_{02}N_o + R_1 + R_2 + R_3}{q_{13}} \quad (10)$$

Note that the solution for the steady state of the concentration of seston in each compartment is zero. This is not surprising since the assumption we made removes the dependence of the concentration of seston on the concentration of TDN. Also, as mentioned with the first model, the concentration of seston had little effect on the concentration of TDN.

Of particular interest is the steady state solution to compartment 3 or the outflow:

$$q_{13}N_3 = q_{02}N_o + R, \quad \text{where } R = R_1 + R_2 + R_3. \quad (11)$$

The change in concentration of TDN at the outflow is equal to the concentrations of each of the possible sources: the inflow and lake sediments. This is a very basic model in that it is simplified to a simple analytic solution. The sum (R) of the terms for the flux of TDN in lake sediments, similar to above, is found to fit the data in 2002 and tested against the data in 2003.

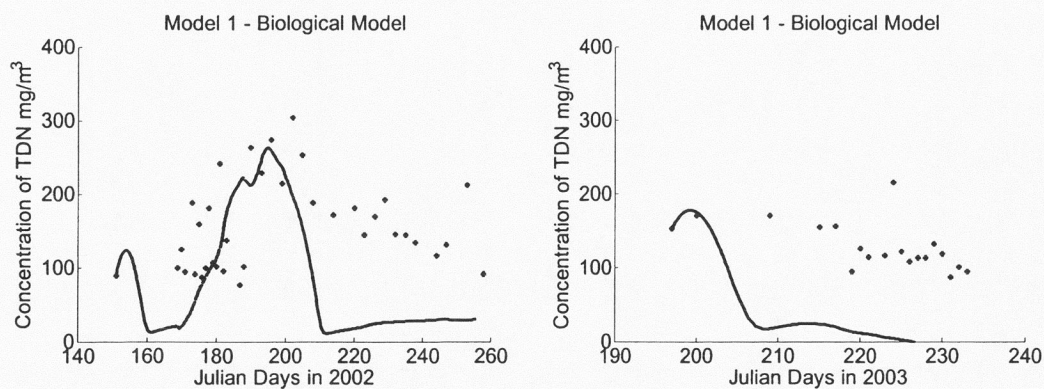
2.8 Model Summary

Each model taken as a whole with the flow model, after determining the entrainment and sedimentation parameters based on outflow and inflow values, takes daily values of the initial velocity of the stream, the temperature of the lake and the inflow, the initial concentration of TDN at the inflow and in the lake, and predicts daily values of the concentration of TDN at the outflow. The results of each of the three models are presented below.

3 Results - Biological Model, TDN Model, and Steady States Model

For each of the three models, the terms of flux of TDN in the lake sediments (R_1 , R_2 , R_3) were used to fit the predicted values of TDN with the actual values of TDN from data in 2002. Each of the three models was then tested against data in 2003. The results for each model including the terms of flux in lake sediments and qualitative analysis are summarized below (Figure 5 and Table 3).

Results of Biological Model



Results of Biological Model - Seston

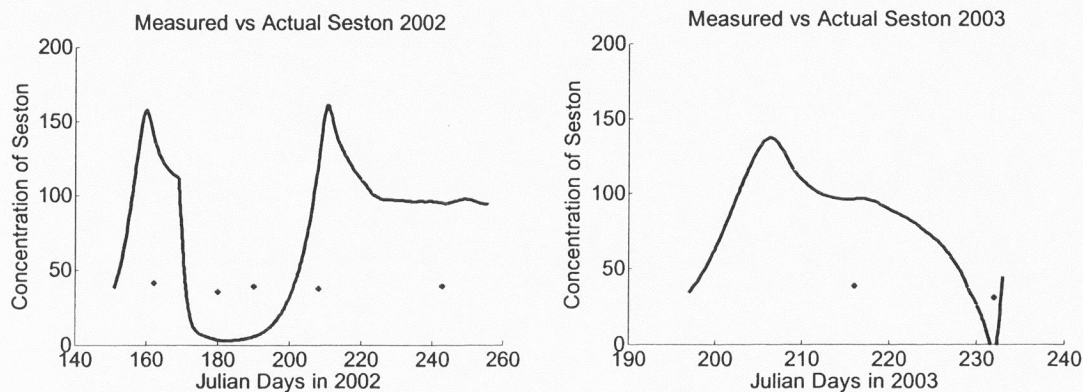


Figure 5a.

Results of TDN Model

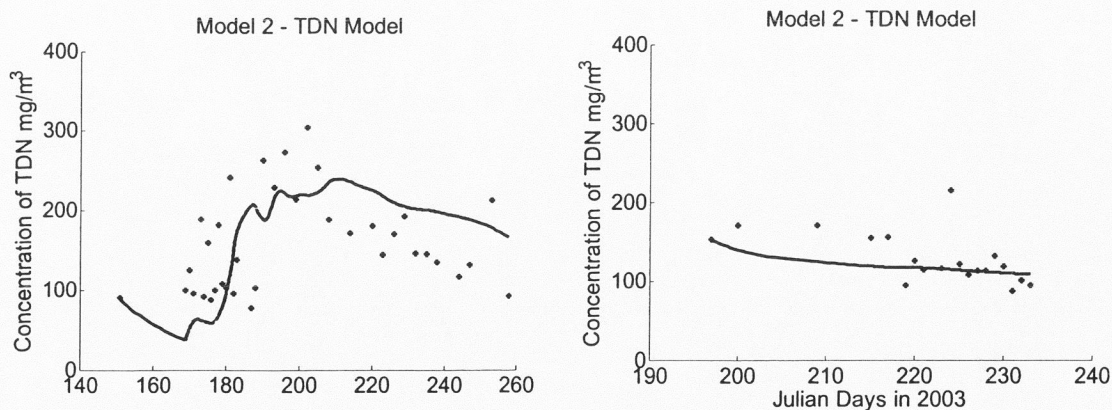


Figure 5b.

Results of Steady State Model

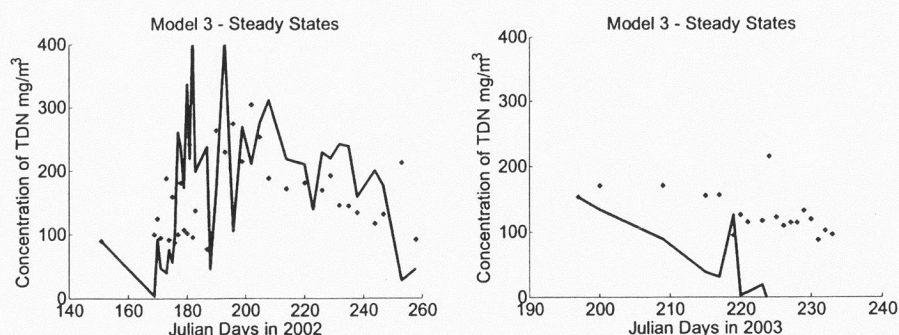


Figure 5c.

Figures 5a-c. The results from each of the three models respectively. Seston was only a parameter in the biological model. Each model was fit to 2002 TDN data and tested against data in 2003. The qualitative results are seen below in Table 3.

Model	Fluxes in Lake Sediments (mg/day)			Net Flux $R_1 + R_2 + R_3$	R^2 Value
	R_1	R_2	R_3		
Biological	-2360697	-1138180	6183765	2684888	0.4734
TDN	3746857	-4081334	-854241	-1188718	0.5478
Steady States				-632635	0.3639

Table 3. The results of the three compartment models. The R^2 value was calculated using the MATLAB function `corrcoef`.

With the value of the sedimentation terms, rough estimates can be made for the values of the flux rate (α_i) from lake sediments for each compartment by dividing by the known area of the compartment contacting the sediment (A_i^s) and measured values of the concentration of TDN in the lake sediment (N_i^s). Having positive and negative values for the sediment parameters is not surprising since measured concentration of TDN at the outflow is sometimes less than the measured value of TDN at the inflow and vice-versa, meaning that a certain amount of the nutrients that enter the lake do not reach the outflow while some nutrients are gained before reaching the outflow. The interpretation of these results is discussed further below.

4 Flow Model

4.1 Hauenstein and Dracos Equations (HDE)

To find the flows used to couple the compartments in the three models above, we solve the system of equations of Hauenstein and Dracos (1984) for the *momentum-dominated* and *buoyancy-dominated*

regions of the inflow into the lake. The flow rates between the remaining compartments are found by assuming conservation of volume.

Where a density difference exists between the inflow and lake water a density current will be generated by the inflow (Fleenor 2000). The density of water increases as temperature decreases (until about 4°C at which point liquid H₂O has maximum density). Solar input warms surface water causing a density difference between the current from the inflow and the lake water; due to the higher density and the slope of the lake bed, the colder inflow plunges beneath the warmer lake water until it reaches a neutral buoyancy at which point it inserts into the lake (see Figure 6 below). Hauenstein and Dracos applied separate integral models to each region based on the steady state mass and momentum balances of the Boussinesq (NSB) approximation (Hauenstein 1984).

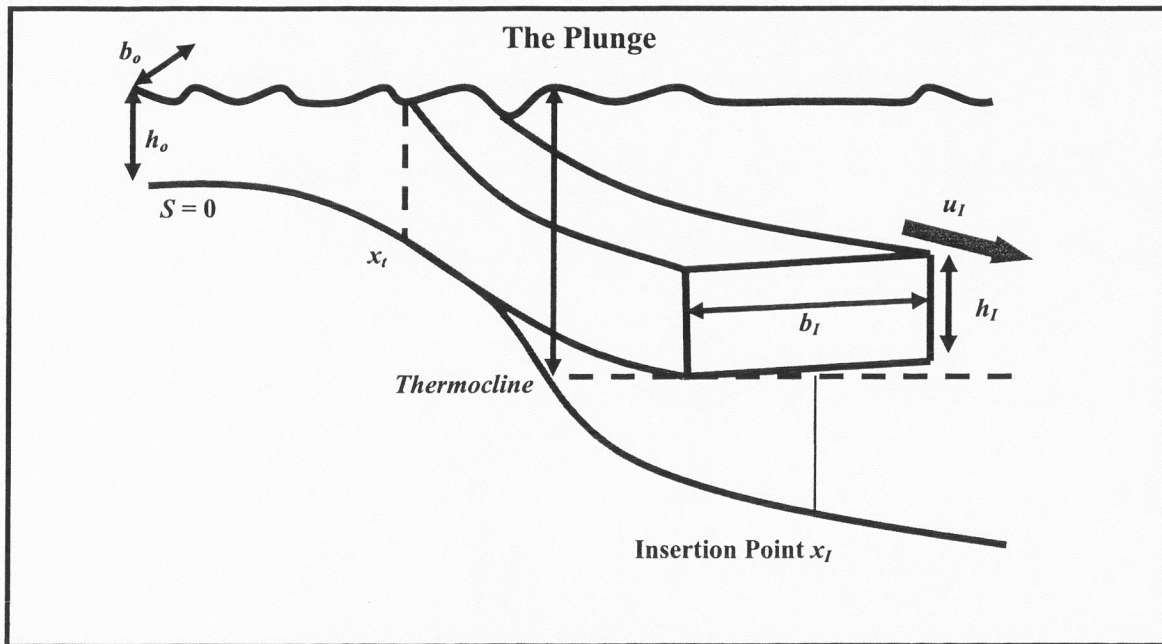


Figure 6. Diagram of the *momentum* and *buoyancy-dominated* regions.

For each region, we assume the density current has the following characteristics: the inflow stream is of fixed height (h_o) and breadth (b_o), with initial velocity (u_o), the lake bed has a fixed slope (S) from the inflow; the lake water is of density (ρ), the current is of density (ρ_c) where $\Delta\rho$ is the difference in density; with the gravity constant (g), and insertion point (x_l). The horizontal entrainment (α_h) is defined by a

constant value relating to buoyant and non-buoyant jets (Turner 1973). There is also a vertical component of entrainment (α_z), and is related to the Richardson number (Fleenor 2000). The flow rate at the inflow ($u_o b_o h_o$) is the inflow (q_{02}) in the compartment models. In both regions there is assumed to be no flux on the boundary with the lake bottom.

Symbols	Description (units)
h	height of plunging current (m)
b	breadth of plunging current (m)
u	velocity of current (m/s)
S	lake Slope (constant)
ρ_c	density of current (mass/m ³)
ρ	density of lake water (mass/m ³)
$\Delta\rho$	$\rho_c - \rho$
α_h	horizontal entrainment rate (constant)
α_z	vertical entrainment rate (constant)
g	gravity constant (m/s ²)
x_t	boundary between momentum and buoyancy dominated regions (m)
x_l	insertion point (m)

Table 4. List of variables and constants used in HDE.

In the *momentum-dominated* region as the stream water is displacing the lake water, the current occupies the entire water column as it covers the initial slope of the lake. The lake water is mixing horizontally on each of the two side boundaries of the inflow. If x is the distance from the inflow, then over a particular distance (dx) in the *momentum-dominated region*, we find the overall change in entrainment to be the area of the boundary ($2h dx$), where the lake water contacts the inflow on both sides, times the velocity of horizontal entrainment ($\alpha_h u$). So we have:

$$d(ubh) = 2\alpha_h uh dx .$$

Thus we have the differential equation for conservation of volume:

$$\frac{d}{dx}(ubh) = 2\alpha_h uh .$$

Assuming that the momentum (ρubh) is conserved we have the equation for the flux of momentum:

$$\frac{d}{dx}(u^2 bh) = 0 \quad (\rho \text{ is constant in this region}).$$

The lake bed slope (S) is assumed to change linearly which gives the equation for the height of the current:

$$h = h_o + Sx.$$

We thus have the equations for the *momentum*-dominated region as defined by Hauenstein and Dracos (1984):

$$\text{conservation of volume} \quad \frac{d}{dx}(ubh) = 2\alpha_h uh \quad (12)$$

$$\text{x-momentum} \quad \frac{d}{dx}(u^2bh) = 0 \quad (13)$$

$$\text{geometric condition} \quad h = h_o + Sx \quad (14)$$

The equations for the *buoyancy*-dominated region as given by Hauenstein and Dracos (1984) are:

$$\text{conservation of volume} \quad \frac{d}{dx}(ubh) = 2\alpha_h uh + 1\alpha_z ub \quad (15)$$

$$\text{x-momentum} \quad \frac{d}{dx}(u^2bh) = \frac{1}{2} \frac{d}{dx} \left(\frac{\Delta\rho}{\rho} gbh^2 \right) - \frac{\Delta\rho}{\rho} gbhS \quad (16)$$

$$\text{y-momentum} \quad \frac{d}{dx}(uvbh) = \frac{1}{2} \frac{\Delta\rho}{\rho} gh^2 \quad (17)$$

$$\text{conservation of buoyancy} \quad \frac{d}{dx} \left(\frac{\Delta\rho}{\rho} gubh \right) = 0 \quad (18)$$

Here the ambient lake water is entrained from both sides of the inflow as well as from a single boundary above. The vertical boundary over a distance (dx), where the lake water is above the inflow, is found by the surface area of the upper boundary of the current ($b dx$) times the velocity of vertical entrainment ($\alpha_z u$). The horizontal entrainment from each side and the vertical entrainment from above give us the above differential equation for volume. The momentum is no longer considered constant. Density differences in both regions can be found using the conservation of buoyancy.

Equations 12-14 are solved analytically for the velocity (u_i), the breadth (b_i), and the height (h_i) at the boundary between the two regions given empirical values for u_o , b_o , h_o , the slope (S), the constant

entrainment value (α_h), and the boundary between the two regions (x_i). The system of equations for the *buoyancy-dominated* region are solved numerically for the velocity (u_i), the breadth (b_i), and the height (h_i) at the insertion point (x_i) given the values of velocity (u_i), breadth (b_i), and (h_i) at the boundary between the two regions, entrainment constant value (α_z), values of lake density (ρ) and the difference in density ($\Delta\rho$) at the boundary, the insertion point (x_i), and certain geometric assumptions on h , which will be outlined below. The sum of the inflow rate (q_{02}) and the entrainment rate (q_{12}) in the compartment model is given by $u_i b_i h_i$.

4.2 Bull Trout Lake-Field Study

In order to test the parameters of the density current included in the flow model, namely the breadth (b), height (h), and velocity (u) of the current, a field study was conducted at Bull Trout Lake (BTL) from June 17 to June 23, 2009. BTL is one of many lakes in the Sawtooth Mountain range located in Central Idaho (Figure 7).



Figure 7. Image of Bull Trout Lake in Central Idaho near Stanley, Idaho.

<http://images.google.com/imgres?imgurl=http://picturesofcascade.com/large7/Bull-Trout_9212L.jpg&imggr>

These lakes reach their peak inflow around mid-June. During peak flow, as part of an extensive study conducted by Montana State and Utah State Universities, rhodamine WT, a fluorescent tracer, was injected into the lake about 50m upstream continuously from June 21st to June 23rd (Figure 8).

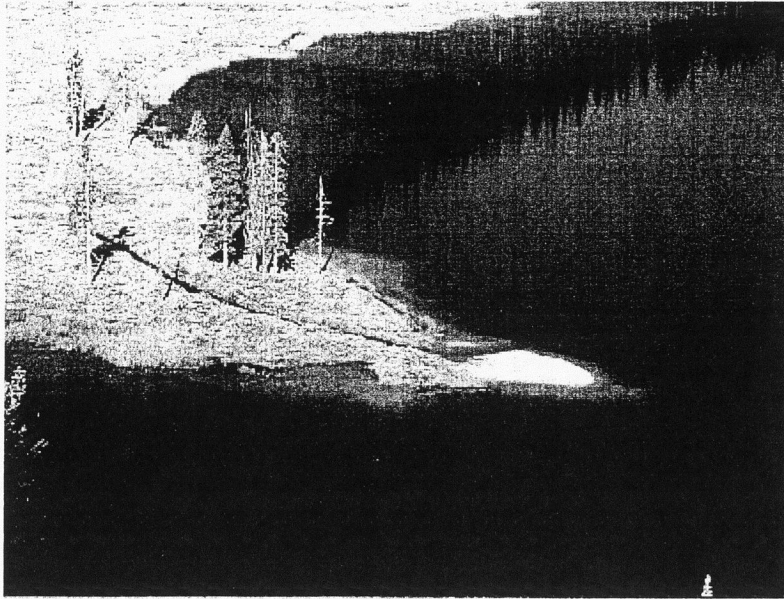


Figure 8. Image of rhodamine being dripped into Bull Trout Lake. Also notice the delta that extends about 25 meters out from the initial inflow. Picture courtesy of Wurtsbaugh et al (Wurstbaugh 2008).

Using a submersible fluorometer, CYCLOPS (Turner Designs, Inc.) voltage was measured to determine rhodamine concentrations throughout the lake. In order to measure the breadth and depth of the plunge into the lake near the inflow, a fence post was placed about 3 meters from the end of the delta that extends about 23 meters into BTL. A rope was connected between the post and the row boat, and the rope made taut so as to stay a consistent distance from the inflow. One person rowed the boat in circular arcs around the inflow while a second person placed the fluorometer at a specific depth and continuously recorded the voltage throughout each arc (Figure 5). Arcs were completed at distances of 15, 30, and 65m at depths of 1, 3, 6, and 9 meters to estimate the breadth (b) of the density current.



Figure 9. Arcs were completed around the inflow using a fluorometer to measure the breadth of the density current. image © 2008 DigitalGlobe, © 2008 Tele Atlas

The breadth (b) was measured by taking the GPS coordinates for the boundaries of high voltage and estimating the length of the arc between them. Depending on the estimated distance to the insertion point, the breadth can vary; the value used for solving the equations is 70 meters (Figure 10).

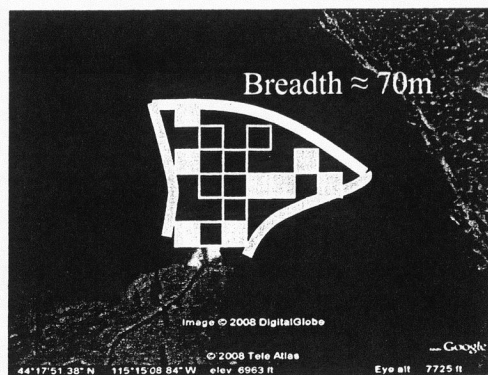


Figure 10. Arcs were completed near the inflow to show the path of the density current and measure its breadth. Red dots indicate peak concentrations of rhodamine, and green dots indicate high concentrations of rhodamine. image © 2008 DigitalGlobe, © 2008 Tele Atlas

In order to estimate the height of the density current, an anchor was used to steady the boat, and the fluorometer was lowered to get a vertical map of the rhodamine concentration in the lake. The height of the density current is estimated by looking at peaks in concentration. The data from the vertical transect were plotted against the depth of the lake (Figure 11). When completing a vertical transect, the

fluorometer contacts the lake bottom, and sediment is disturbed creating outlying concentrations not due to rhodamine. These concentrations are ignored in the measurement. Depending on the insertion point, the height of the peak concentration may vary; at the estimated insertion point the value of h was measured to be 2 meters.

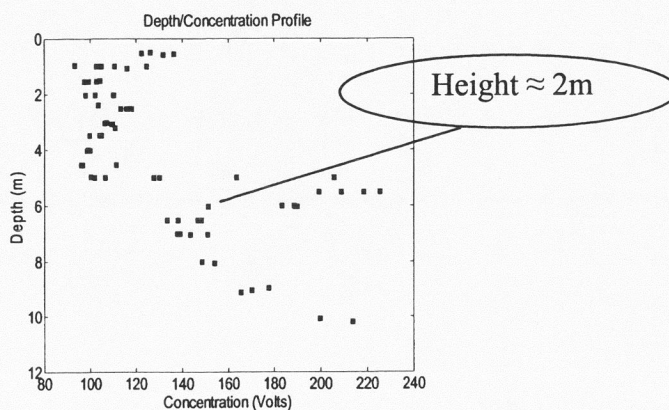


Figure 11. A vertical transect was taken to measure the height of the density current. The second peak represents the fluorometer displacing lake sediment as it contacts the lake bottom and is ignored in the height measurement.

As mentioned with the breadth and the height, the location of the insertion point (x_i) of the density current is critical to the flow model. The density current depends on the density of the water in the current which in turn depends on the temperature of the water. As warmer lake water is entrained into the colder density current, the temperature of the current rises. At the same time, the lake water grows colder as the current plunges deeper. When the temperature of the current matches the temperature of the lake, the density will equalize and the current will insert into the lake. The depth at which this occurs is assumed to be the thermocline of the lake. Throughout the lake, vertical temperature transects were taken to estimate the depth of the thermocline (Figure 12).

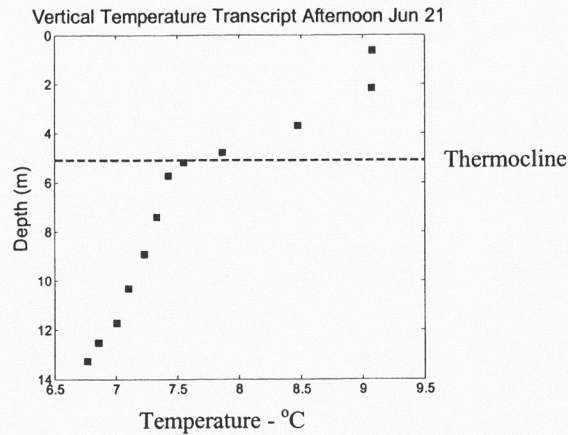


Figure 12. A vertical transect of the lake temperature on June 21, 2009. The thermocline is measured at 5m.

Measurements of the thermocline vary at different points in the lake and at different times during the day. For instance, in the afternoon near the inflow (within 100 m), the thermocline was measured near the surface to be 1.5 to 2 meters. Early in the morning near the inflow (within 100 m), the thermocline was measured deeper to be 5 to 6 meters (Figure 12 above). In the lake models, the middle layer or compartment is defined to encompass the upper and lower bounds for the measured values of the insertion point.

With the depth of the thermocline measured and the assumption that the plunge inserts near the thermocline the point of insertion (x_i) is measured using lake morphometry (Figure 13).

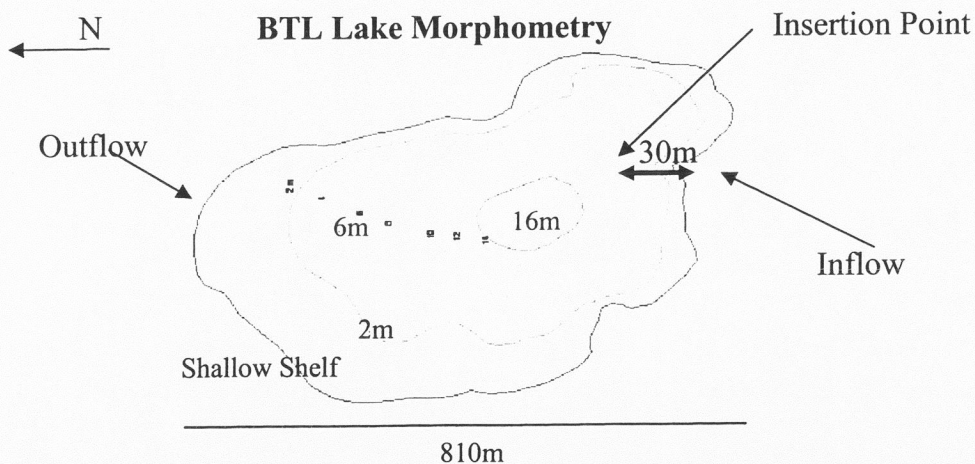


Figure 13. Contour map of BTL. The lake flow from south to north, has a large shallow shelf along the northwestern side of the lake, a delta that runs about 23 meters into the lake, and a maximum depth of about 16 meters.

The estimated depth of the bottom of the middle compartment (6m) was located on the map, and the distance from the inflow to the first marker of 6m in the direction of the density current was calculated. The insertion point was measured to be 30 meters.

The boundary between the momentum-dominated and buoyancy-dominated regions (x_l) was determined by observing the flow of rhodamine into the lake. At BTL, from the inflow extends a small delta about 23 meters into the lake (Figure 8). During high flow in the summer, the water flows over this delta before reaching the initial slope of the lake bed. The density current was observed to plunge very near the end of the delta at the first major change in the slope of the lake. From this observation, the boundary between the two regions was determined to be 23 meters.

The height of the current over the delta was relatively flat and was measured to be a constant $h = 0.2$ m for $0 \leq x \leq 23$. The slope (S) of the lake is 0 over the delta. The slope of the lake over the buoyancy-dominated region (0.25) was measured by taking the depth at the insertion point and the depth at the end of the delta. The initial breadth of the current near the inflow at the beginning of the delta was measured to be 5 meters. The velocity of the density current (u) was measured at the inflow using a flow meter. Near the inflow at the location of the plunge, measurements could only be taken from the boat. Due to the unsteadiness of the boat an accurate measurement of the velocity at different depths was not possible. Thus the velocity of the current at the insertion point (u_l) could not be measured directly. It was observed, however, rhodamine had crossed the entire length of the lake within 24 hours of release. The length of the lake was measured to be approximately 810 meters, and dividing this by one day we have:

$$\frac{810m}{1day} = \frac{8 \times 10^4 cm}{8.64 \times 10^4 sec} = \frac{8cm}{8.64 sec} \approx .93cm/sec.$$

This is the approximate minimum velocity required for the current to traverse the entire lake in one day and is a rough estimate of the minimum velocity at the insertion point (u_l). This was used as an estimate of the value of the velocity at the insertion point (u_l). Values used in calculations are summarized in the table below and shown in the figure below.

Symbols	Description
$h_o = 2 \text{ cm}$	height of the current over the delta ($0 \leq x \leq x_t$)
$b_o = 5 \text{ m}$	initial breadth of density current
$S = 0$	lake Slope over delta ($0 \leq x \leq x_t$)
$S = .25$	lake Slope from delta ($x_t \leq x \leq x_I$)
$x_t = 23 \text{ m}$	boundary between momentum and buoyancy
$x_I = 30 \text{ m}$	dominated regions
$b_I = 70 \text{ m}$	insertion point determined by thermocline
$h_I = 2 \text{ m}$	breadth of current at insertion point
$u_I = .93 \text{ cm/s}$	height of current at insertion point
	velocity of current at insertion point

Table 5. Measured values for HDE parameters and boundary conditions at BTL.

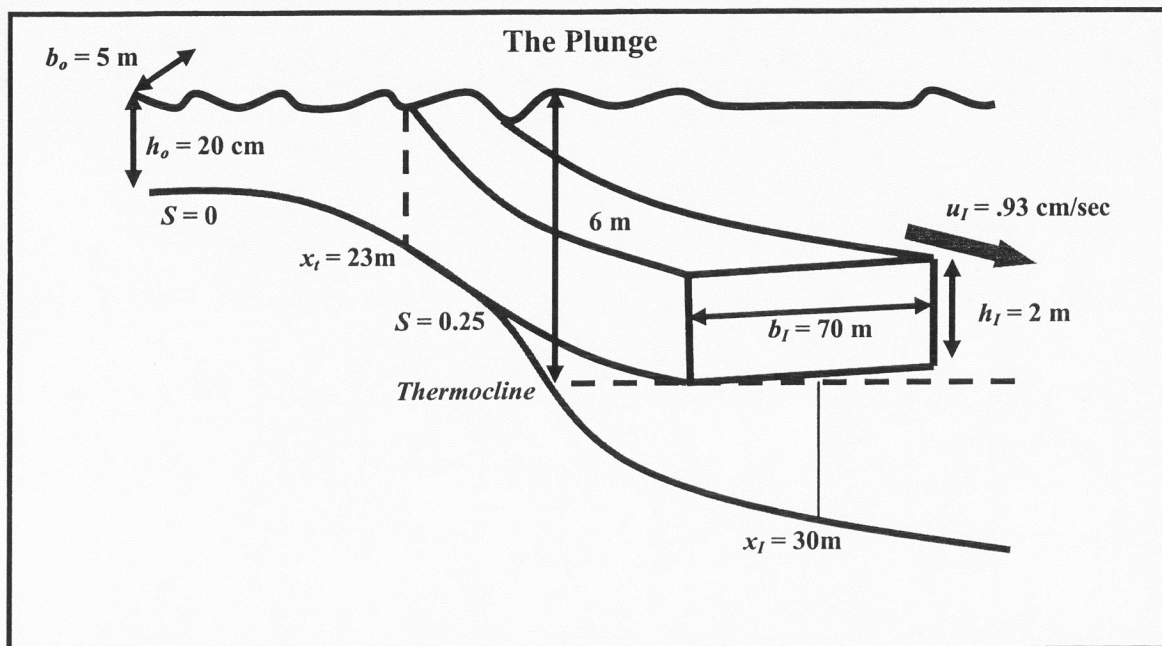


Figure 14. A graphical depiction of the plunge into Bull Trout Lake. Measurements were taken from field data collected in the study in 2008.

The initial velocity (u_o) of the inflow is determined from daily discharge data taken near the inflow (Arp 2006). The initial velocity is found by taking the discharge (m^3/sec) and dividing by the initial breadth (b_o) and the initial height (h_o). The initial density of the current ($\rho_{C,o}$) is determined from temperature data at the inflow (Wurtsbaugh 2008). Values of water density based on temperature were taken from a table of values for distilled water (Walker 1998). From these values, a quadratic formula was fit to the data using a build in MATLAB fitting tool:

$$\rho = (-6.3 \times 10^{-6})T^2 + (3.5 \times 10^{-5})T + 1.$$

Using this relation, the lake and current densities were calculated. The initial density differential used in the flow model (equations 16–18 $(\Delta\rho/\rho)_o$) was found by taking the difference of the initial densities of the lake and inflow and dividing by the initial density of the lake taken at the bottom of compartment 1. The differential in density was then calculated as a variable in the model.

4.3 Solution to Hauenstein and Dracos Equations for BTL

4.3.1 Solution in Momentum-Dominated Region

With the initial values of h_o , b_o , S , the plunge point (x_i), and the initial velocity (u_o) determined from discharge data, the system of equations for the momentum-dominated region is solved analytically. In the geometric condition (Equation 14), the slope (S) is zero over the delta, so we have the height as a constant:

$$h = h_o = 0.2\text{m}. \quad (8)$$

Using equations 13 and 14, we integrate both sides with respect to x to find:

$$u^2 b = u_o^2 b_o.$$

The height $h = h_o$ divides out. We then solve this for b to find:

$$b = \frac{u_o^2 b_o}{u^2}. \quad (9)$$

This is substituted into the equation for the conservation of volume (12):

$$\frac{d}{dx}(ubh) = 2\alpha_h u h \rightarrow \frac{d}{dx}\left(u \frac{u_o^2 b_o}{u^2}\right) = 2\alpha_h u \rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = \frac{2\alpha_h u}{u_o^2 b_o}.$$

Using separation of variables, we integrate (12) to find:

$$\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{2\alpha_h u}{u_o^2 b_o} \rightarrow -\frac{1}{u^2} \frac{du}{dx} = \frac{2\alpha_h u}{u_o^2 b_o} \rightarrow -\frac{1}{u^3} du = \frac{2\alpha_h}{u_o^2 b_o} dx \rightarrow \frac{1}{2u^2} = \frac{2\alpha_h}{u_o^2 b_o} x + C.$$

The constant is determined by using the initial values at $x = 0$ to find:

$$\frac{1}{2u^2} = \frac{2\alpha_h}{u_o^2 b_o} x + C \rightarrow \frac{1}{2u^2} = \frac{2\alpha_h}{u_o^2 b_o} x + \frac{1}{2u_o^2} \rightarrow u = \sqrt{\frac{u_o^2 b_o}{4\alpha_h x + b_o}}.$$

Substituting this back into equation (13) to solve for b , we have the solution to the system of equations in the momentum-dominated region:

$$h = 0.2\text{m} \quad (14)$$

$$b = 4\alpha_h x + b_o \quad (19)$$

$$u = \sqrt{\frac{u_o^2 b_o}{4\alpha_h x + b_o}} \quad (20)$$

The flow rate (m^3/s) of lake water being entrained in the momentum-dominated region (Q_{md}) can be found by subtracting the initial flow rate from the flow rate at the plunge point:

$$Q_{md} = u(x_t)b(x_t)h(x_t) - u(x_o)b(x_o)h(x_o) = u_t b_t h_t - u_o b_o h_o.$$

The value Q_{md} will later be added to the rate of entrainment in the buoyancy-dominated region (Q_{bd}) to determine flow between compartment 1 and 2 (q_{12}) in the compartment models.

The values of u_t , b_t , and h_t at the plunge point (x_t) are also initial conditions for the buoyancy-dominated region. The value of the entrainment rates α_h and α_z are determined given the measured values of u_t , b_t , and h_t at the insertion point.

The variable of density ($\Delta\rho/\rho$) in the momentum-dominated region is determined from the conservation of buoyancy (18). Integrating both sides and solving for $\Delta\rho/\rho$ we have:

$$\frac{d}{dx} \left(\frac{\Delta\rho}{\rho} gubh \right) = 0 \rightarrow \frac{\Delta\rho}{\rho} ub = C.$$

The constants g and h are independent of x and divide out of the equation. With the initial conditions at $x = 0$ we solve for the constant C and determine the variable of density:

$$\frac{\Delta\rho}{\rho} ub = C \rightarrow \frac{\Delta\rho}{\rho} ub = \left(\frac{\Delta\rho}{\rho} \right)_o u_o b_o \rightarrow \frac{\Delta\rho}{\rho} = \left(\frac{\Delta\rho}{\rho} \right)_o \frac{u_o b_o}{ub}.$$

From this we can find the variable of density at the plunge point $((\Delta\rho/\rho)_t)$, which is an initial condition for the buoyancy-dominated region.

4.3.2 Solution in Buoyancy-Dominated Region

With the values at the plunge point (x_i) of the slope (S), the height (h_i), the breadth (b_i), the velocity (u_i) and the variable of density ($(\Delta\rho/\rho)_i$), we solve the system of equations 15-18 in the buoyancy-dominated region. In the four equations, we have five variables: u , v , b , h , and $\Delta\rho$. The vertical velocity (v) is found only in equation 17. The other variables are therefore independent of v , so equation 17 is not needed to determine the other parameters, and the system reduces to equations 15, 16, and 18.

That still leaves four variables with just three equations, so to solve the system, we assume that the height of the current grows in a smooth manner from 20cm on the delta (x_i) to 2m at the insertion point (x_I):

$$h(x) = \frac{C_1}{x} + C_2, \text{ where } C_1 \text{ and } C_2 \text{ are constants.}$$

The constants C_1 and C_2 are determined by these values, that is the height and distance at the plunge point ($h_i = 0.2\text{m}$ and $x_i = 23\text{m}$) and the height and distance at the insertion point ($h_I = 2\text{m}$ and $x_I = 30\text{m}$).

We have:

$$h_i = \frac{C_1}{x_i} + C_2 \quad \text{and} \quad h_I = \frac{C_1}{x_I} + C_2.$$

Plugging in the values and solving for C_2 we have:

$$.2 = \frac{C_1}{23} + C_2 \rightarrow C_2 = .2 - \frac{C_1}{23}.$$

Plugging this into the equation at the insertion point we have:

$$2 = \frac{C_1}{30} + C_2 \rightarrow 2 = \frac{C_1}{30} + .2 - \frac{C_1}{23} \rightarrow C_1 \approx -177.51.$$

From this we find:

$$C_2 = .2 - \frac{C_1}{23} \approx 7.92.$$

We thus have the estimated height of the current in the buoyancy-dominated region:

$$h(x) = \frac{-177.51}{x} + 7.92.$$

With this geometric assumption, the non-linear system of differential equations for the buoyancy-dominated region (equations 15, 16, and 18) is solved numerically using the built in MATLAB differential equation solver ode15s. Differentials in each equation are expanded using the product rule to obtain:

$$\frac{du}{dx}bh + u\frac{db}{dx}h + ub\frac{dh}{dx} = 2\alpha_h uh + 1\alpha_z ub$$

$$2u\frac{du}{dx}bh + u^2\frac{db}{dx}h + u^2b\frac{dh}{dx} = \frac{1}{2}\frac{d}{dx}\left(\frac{\Delta\rho}{\rho}\right)gbh^2 + \frac{\Delta\rho}{\rho}\frac{db}{dx}(gh^2) + \frac{\Delta\rho}{\rho}gb2h\frac{dh}{dx} - \frac{\Delta\rho}{\rho}gbhS$$

$$\frac{d}{dx}\left(\frac{\Delta\rho}{\rho}\right)gubh + \frac{\Delta\rho}{\rho}\frac{du}{dx}gbh + \frac{\Delta\rho}{\rho}u\frac{db}{dx}gh + \frac{\Delta\rho}{\rho}gub\frac{dh}{dx} = 0$$

The system is put into matrix form for the solver to run:

$$\begin{bmatrix} bh & uh & 0 \\ 2ubh & u^2h - \frac{1}{2}\frac{\Delta\rho}{\rho}gh^2 & -\frac{gbh^2}{2} \\ \frac{\Delta\rho}{\rho}gbh & \frac{\Delta\rho}{\rho}guh & gubh \end{bmatrix} \begin{bmatrix} \frac{du}{dx} \\ \frac{db}{dx} \\ \frac{d}{dx}\left(\frac{\Delta\rho}{\rho}\right) \end{bmatrix} = \begin{bmatrix} 2\alpha_h uh + \alpha_z ub \\ \frac{\Delta\rho}{\rho}gbhh' - u^2bh' - \frac{\Delta\rho}{\rho}gbhS \\ -\frac{\Delta\rho}{\rho}gubh' \end{bmatrix}.$$

Solving these equations with known α_h and α_z , we obtain the velocity (u), breadth (b), and variable of density ($\Delta\rho/\rho$) over the buoyancy-dominated region ($x_l \leq x \leq x_l$). The flow rate (m^3/s) of lake water being entrained in the buoyancy-dominated region (Q_{bd}) can be found by subtracting the flow rate at the plunge point from the flow rate at the insertion point:

$$Q_{bd} = (ubh)(x_l) - (ubh)(x_l) = u_l b_l h_l - u_i b_i h_i.$$

The sum of the two entrainment rates ($Q_{md} + Q_{bd} = q_{12}$) is the flow rate between compartment 1 and compartment 2 of the three lake models.

Known values for α_h and α_z and initial conditions for the velocity (u_o), breadth (b_o), and height (h_o) can be used to predict the values at the insertion point (u_i , b_i , and h_i respectively). Instead, we have known values for u_i , b_i , and h_i taken from the field study, and we can use these to determine the values of α_h and α_z .

4.4 Eigenvalue Problem for α_h and α_z

The measurements found at the insertion point ($x_I = 30\text{m}$) of the breadth ($b_I = 70\text{m}$), and the velocity ($u_I = .93\text{cm/s}$) can now be used to determine the values for α_h and α_z , that is, we solve the problem:

$$u(\alpha_h, \alpha_z) - u_I = 0$$

$$b(\alpha_h, \alpha_z) - b_I = 0$$

Put simply we find α_h and α_z , that fit the solution of the flow model to the actual measured values of the breadth and the velocity. We therefore have two free variables and two conditions to solve making this an eigenvalue problem. To find α_h , and α_z we minimize the sum of the squares of the errors between the actual and measured values for b and u at the insertion point.

$$E(\alpha_h, \alpha_z) = (u(\alpha_h, \alpha_z) - u_I)^2 + (b(\alpha_h, \alpha_z) - b_I)^2.$$

This is done by using the built-in MATLAB function `fminsearch`. A function for the solution to equations 12-15, 17 and 18 is written including the momentum and buoyancy-dominated regions. The input values for `fminsearch` are the function for the solution with initial values at the inflow and a variable representing the two entrainment rates and the output is the error above. The MATLAB function `fminsearch` then finds the optimal choices of α_h and α_z to minimize the error. The basic code is given below (Figure 15). The measurements above were taken near peak flow in late June. The initial values were taken from June and July in 2002 (Table 6). Taking the average of the calculated α_h 's and α_z 's, we have $\alpha_h = .0095$ and $\alpha_z = .0101$. These rates are much smaller than those found in literature values; $\alpha_h = .1$ as defined by Turner (1973). With these two entrainment rates, the solution to the flow model can be found to give the flow rate at the insertion point ($q_{21} = u_I b_I h_I$) and the flow rate of entrainment ($q_{12} = Q_{md} + Q_{bd}$). These values are used in the biological compartment model above.

```

[[alphah alphaz],minerror,ef] = fminsearch(@(x) HDE([initial values],x),[guess])

function error = HDE([initial values],x)
[X Y] = ode15s(b,[interval],[Values at Plunge Point],A);
u = Y(end,1)
b = Y(end,2)
error = (u - .0093)^2 + (b - 70)^2

```

Figure 15. Code for finding α_h, α_z that minimize the error between the predicted values of u_l and b_l from the model and the actual values measured from field data ($u_l = .0093\text{m/s}$ and $b_l = 70\text{m}$). Initial values can be found in figure 9. A guess is made for the values of α_h and α_z as a starting point for the matlab function. An exit flag (ef) is given to indicate if the MATLAB function found a minimum or reached its set limit of calculations.

JD 2002	Input Values			Results		$E(\alpha_h, \alpha_z)$
	u_o (m/s)	LTemp (°C)	STemp (°C)	α_h	α_z	
151	0.10378	5.07375	5.39667	0.009095	0.010532	1.44E-05
171	1.30099	8.04659	5.96333	0.009219	0.010415	0.010154
176	0.752	8.10197	6.105	0.009243	0.010412	0.002952
214	0.497	12.90089	7.10667	0.010072	0.010089	0.001086

Table 6. Results of solving for α_h and α_z , where JD is Julian day, LTemp is the temperature of the lake, STemp is the temperature of the inflow, and E is the total entrainment rate. Input data taken from discharge data in 2002. Alphas are found using the built in MATLAB function fminsearch.

4.5 Flow Model Summary

In solving the Hauenstein and Dracos Equations (1984), flow rates of entrainment and flow rates at the insertion point were found using physical data taken at the field study in 2008. As mentioned above in determining the breadth and the depth of the plunge, the insertion point is critical. Several methods were used to calculate the insertion point. The insertion point was left free in the flow model and was approximated by placing tolerances on the temperature and velocity and finding distance at which the model reached these values. Small changes of the insertion point in the model yielded dramatic differences in the breadth of the flow due to the rapid growth of the breadth from the inflow. However, when coupling the breadth with the velocity, which was decreasing, the overall flow at the insertion point was not as sensitive to changes in the insertion point. For example, insertion point changes of ± 5 meters altered flow rates by approximately 0.05%. Therefore, the compartment models for TDN are not sensitive to minor changes in the insertion point.

5 Discussion

From the results of the three compartment models, we see that the second model, the TDN model, produced the best fit. It reasonably predicts the concentration of TDN at the outflow given the concentration of TDN at the inflow. This shows that the physical structure of the lake is significant in predicting nutrient cycling through mountain lakes. As with each of the models, it is run on intervals between measured inflow values for TDN. More ample data for the concentration of TDN at the inflow produces more precise predictions for the outflow. Some fluctuations occur in the data causing some bias, like a fly getting caught in the sample. This model also shows that a stratified lake structure in a compartment framework can reasonably predict the flow of nutrients through mountain lakes without adding a great deal of complexity.

The first two models were fit to the data using the parameters R_1 , R_2 , and R_3 and the third model was fit using the sum of the three values. These parameters are roughly defined as the nutrients gained and lost from lake sediments. In compartment 1, positive and negative values could be gains or losses in TDN from the sides of compartment 1 including small lake inlets and groundwater. In compartment 2, negative values could be the initial loss of sediment that sinks soon after the inflow enters the lake. Positive and negative values in compartment 3 could be gains from groundwater and inlets and losses to the lake bottom as the slope of the lake rises to the outflow. The results from the second model suggest that the study lake, BTL, is a sink since the sum of the fluxes in concentrations of TDN within the lake is negative (Table 3).

The initial choice of adding each term in all compartments was a natural one since the area of each compartment intersecting the lake bottom was given in the lake morphometry. However, the loss to and gain of nutrients could be isolated to two or even one of the three compartments. The first model, the biological model, although more complex failed to yield better results. The dynamics of seston could not be accurately modeled. The model was tested a number of times with different initial values, and although some of the dynamics of the changes in actual seston could be achieved, the model never

followed observations. Basic parameterization, that is modifying the initial input values for the constants in the biological model (growth rates, death rates, etc.) yielded similar results.

The third model, the steady states model, although showing some good results in fitting the data in 2002, lacked predictive power when tested against the data in 2003. The appeal to this model is the simplicity of the calculation and with a more precise compartment structure, that is creating a suite of more precise compartments, this model could show better results. From these models we see that higher complexity does not often yield better results, especially when data for the higher complexity model is not available. However, removing all complexity can make it difficult for dynamics to be modeled consistently over a long period of time. .

6 Conclusion

The structure of mountain lakes is significant in relating input and output values of TDN. An intermediate complexity mathematical model for nutrient cycling in mountain lakes can include the dynamics associated with the stratification of mountain lakes without a great increase in complexity. The model we present reasonably predicts the change in concentration of TDN that occurs between the inflow and outflow at BTL. The differential equations constructed by Hauenstein and Dracos (1984) do a reasonable job of predicting the flow rate of the density current associated with mountain inflows. The multiple compartments for the lake model are able utilize the structure of lakes to isolate particular biological processes to certain regions of the lake, like seepage from lake sediments, thus including the added boundary of the lake floor. These models are solved with lower complexity than more computationally intensive models. This shows the robustness of a well-mixed container and gives an example of how the structure of a lake can be utilized to include specific parameters associated with nutrient uptake while retaining the robustness the well-mixed model.

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